Out-of-equilibrium properties and non-linear effects for interacting quantum impurity systems in their strong coupling regime

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We build an exact description of out-of-equilibrium fixed points in quantum impurity systems, that is able to treat time-dependent forcing. We then show that exact analytical out-of-equilibrium results can be obtained in interacting quantum impurity systems in their strong coupling regime, provided they are integrable *at* equilibrium and they only allow for integer charge hopping at low energy. For a large class of local operators, including currents (of charge, energy, ...), we show that out-of-equilibrium correlators can be obtained as equilibrium correlators of effective operators. We explicitly show that out-of-equilibrium properties are universal, i.e. they depend on the microscopic details through a *single* energy scale. We apply our approach to the Interacting Resonant Level model, and obtain the expansion of the universal scaling function for the charge current as a function of voltage, temperature, and frequency, up to order seven.

The description of out-of-equilibrium properties of interacting quantum many-body systems is certainly a major challenge of modern physics. Amidst the immensely large class of such systems, quantum impurity systems (QIS) are remarkable in that they can be considered as the simplest ones: they are defined as (or can be brought to) one-dimensional systems homogeneous in space (metallic quantum wires that play the role of baths), except at one point where the interaction is concentrated. Moreover, they can easily be forced out-of-equilibrium by coupling the wires to different external, possibly time-dependent, sources (of charge, energy,...).

QIS model transport through microscopic (e.g. quantum dots [1, 2]) or nanoscopic (e.g. atoms or molecules[3–5]) objects connected to macroscopic electrodes, a rich experimental field driven amongst other things by long term efforts towards the miniaturization of electronics. Understanding transport in QIS is thus of crucial importance. Whereas linear transport – that boils down to equilibrium properties – is fairly well under control theoretically, non-linear effects are significantly harder to predict: solving the out-of-equilibrium theory of QIS unfortunately turns out to be a considerable problem, mixing many-body aspects (interactions make it complicated) with the intrinsically open geometry of the out-of-equilibrium problem.

The one-dimensional character of QIS comes along with a realm of powerful methods – be they analytical or numerical – that gives the hope one could solve the out-of-equilibrium problem. A generic feature of QIS at equilibrium is that the impurity/wire coupling, no matter how small it is, has drastic consequence on the ground-state of the system: at temperature $T = 0$, properties are described by a strong coupling (SC) fixed point[6–8], that physically corresponds to the hybridization of the wires and governs the low voltage behavior: for example, for systems with a Fermi liquid SC fixed point[9, 10], hybridization results in a linear $I(V)$ characteristic at small voltage, with the conductivity $G_0 = \frac{\partial I}{\partial V}\big|_{V=0}$ being maximal at $T = 0$. A typical energy scale, called here the hybridization temperature T_B (or equivalently a hybridization length $\lambda_B = \frac{\hbar v_F}{k_B T_B}$ with v_F the Fermi velocity) and akin to a Kondo temperature, marks the crossover to the SC regime: as soon as E/T_B acquires a finite value, E being any energy scale at which the system is probed, additional many-body scattering mechanisms must be taken into account to describe the physics – and, incidentally, non-linear effects. This SC regime, defined by all energy scales being smaller than T_B, is the focus of this work (Fig.1) ; it is not captured by conventional perturbative expansions, performed around the weak coupling (WC) fixed point, that encounter convergence problems when the largest [42] of the ratios E/T_B becomes $\lesssim 1$.

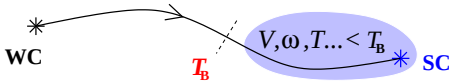

FIG. 1: A sketch of the renormalization group trajectory of the QIS interpolating between the decoupled (WC) and hybridized (SC) fixed points. The energy scale T_B marks the crossover between the weak coupling regime and the SC regime (our focus, shaded area).

Numerical techniques have made significant progresses recently, with the development of efficient time-dependent algorithms. Nevertheless, real-time numerical approaches performed on finite systems like time-dependent DMRG face a difficulty in extrapolating to large sizes or times [11], and methods formulated in infinite systems like diagrammatic Monte Carlo [12] or time dependent NRG [13] run into problems for accessing the stationary state in the SC regime where the equilibration time becomes too large. Convergence of real-time renormalization group [14] is problematic in the SC regime.

On the analytical side, since many QIS are integrable (and thus exactly solvable) at equilibrium, one could hope for a full exact solution out-of-equilibrium. While this is in fact the case for free QIS (by free we mean

that the hamiltonian is quadratic in fermions, a very special subset of integrable QIS) where the Landauer Buttiker formalism applies[15–17], and also for some interacting QIS that are amenable to free QIS by possibly complicated, non local transformations[18–21], this might well be a property of free fermion systems. Indeed, there are to the best of our knowledge only two examples of *interacting* QIS that could be solved exactly out-of-equilibrium using integrability: the Boundary Sine Gordon model, solved by algebraic methods [22] or by Thermodynamical Bethe Ansatz (TBA) [23, 24], and the Interacting Resonant Level model (IRLM) at its self dual point using TBA [11, 25]. It is likely that the integrable structure is not preserved when the system is coupled to reservoirs, except in a few exceptional cases. Moreover, the exact TBA approach is limited to static forcing, and cannot treat e.g. the experimentally relevant AC regime.

In this Letter, we are going to use integrability in a weaker sense, by showing that it can provide an *exact*, *universal*, and *systematic* expansion of physical quantities in out-of-equilibrium conditions (including time-dependent forcing) in the SC regime, that can be pushed to arbitrary order in principle. Precisely, we compute the universal scaling functions determining the physical quantities (e.g. $I = V\, f(\frac{V}{T_{\rm B}}, \frac{T}{T_{\rm B}}, \frac{\omega}{T_{\rm B}}, ...)$ for the current) in an expansion in $T_{\rm B}^{-1}$, building a kind of Sommerfeld expansion in interacting systems. We implement our approach in Fermi liquids, but it goes far beyond the standard Fermi liquid (FL) approach [9, 10] that only fixes the first correction to the $T = 0$ linear regime: indeed, at orders higher than $T_{\rm B}^{-2}$, *new* scattering processes, not considered in the FL approach, come into play and affect non-linearities in the $I(V)$ characteristics. In contrast to what happens at higher dimensions, [26, 27] we find that physical quantities are analytic.

We proceed in three steps: First, the forcing out of equilibrium (that can be dynamical) is exactly incorporated in the description of the SC fixed point for arbitrary QIS: we build the Hershfield's operator [28] that determines the out-of-equilibrium density matrix.

Second, we focus on integrable QIS and carry on a Keldysh expansion in the distance to the SC fixed point. In the case of a super Fermi liquid (to be defined below), analytical properties allow for an *exact* expression for the out-of-equilibrium average value of local operators $\mathcal{A}(x,t)$ in terms of *equilibrium* average value of effective operators in a free theory $H_0^{\rm SC}$:

$$\langle \mathcal{A}_1(x_1,t_1)...\rangle_{\rm N.Eq} = \left\langle \mathcal{A}_1^{\rm eff}(x_1,t_1)...\right\rangle_{H_0^{\rm SC}} \qquad (1)$$
$$\mathcal{A}^{\rm eff}(x,t) = \mathscr{U}_{\rm N.Eq}\cdot\mathscr{U}_{\rm B}\cdot\mathcal{A}(x,t)$$

where the operators undergo two operations : a dressing $\mathscr{U}_{\rm B}$ by all scattering processes with the impurity, and a gauge transformation $\mathscr{U}_{\rm N.Eq}$ that translates the out-of-equilibrium forcing. We give in Eqs.(4,5) the explicit form of these super-operators.

Third, we obtain an explicit expression for the effective operators (1) in a systematic expansion around the SC fixed point, that organizes as a series in integer powers of $\frac{E}{T_{\rm B}}$, where E is any energy scale in the problem: voltage, frequency, temperature(s)... We carry out this expansion at order $(\frac{E}{T_{\rm B}})^{-7}$ summing up ~ 1500 diagrams using a Mathematica code. In full generality, the value of the radius of convergence of such an expansion is an open question. In the case of the IRLM investigated below, the radius is finite when there are no interactions, even in the case of harmonic forcing[29, 30], and also at the self dual point in the static limit [25], so that it gives good confidence that it remains finite for any value of the interaction at least for harmonic forcing. It might be 0 in other situations: our expansion would then be an asymptotic series.

We now implement this method in the concrete example of the IRLM [31], that has recently earned the status of benchmark in out-of-equilibrium physics in QIS[11, 25, 32, 33] as one of the simplest QIS allowing for non equilibrium in the presence of interactions. The IRLM consists of two baths of free spinless electrons coupled via tunnel hopping to a single level, that interacts capacitively with the electrodes via a short range potential with strength U. After the standard steps of linearizing around the Fermi points and unfolding [34], the two semi-infinite wires are described by two right-moving free fermionic fields $\psi_{1(2)}(x,t)$ coupled at $x=0$ to the impurity level (with creation operator $d^\dagger$):

$$H = \sum_a H_0[\psi_a] + H_{\rm B},\quad H_0[\psi] = -iv_{\rm F}\int_{-\infty}^{\infty} dx\,\psi^\dagger\partial_x\psi \quad (2)$$
$$H_{\rm B} = \gamma_a\psi_a^\dagger(0)d + \text{h.c.} + U{:}\psi_a^\dagger\psi_a{:}(0)\big(d^\dagger d - \tfrac{1}{2}\big) + \epsilon_d\, d^\dagger d,$$

A sum over $a = 1, 2$ is implied in $H_{\rm B}$, and in the following we set $v_{\rm F} = e = \hbar = k_B = 1$. For $U = 0$, one recovers a free theory, the Resonant Level Model. In the interacting model $U \neq 0$, at equilibrium, standard manipulations (see e.g. Ref.[35]) lead to the conclusion that the tunneling term has a scaling dimension $D(U) = \frac{1}{4} + \big(\frac{\arctan(\frac{2}{U} - \frac{U}{2})}{\pi}\big)^2$, implying that it is relevant ($D < 1$) in the entire repulsive regime $U > 0$ and part of the attractive regime. The free theory corresponds to $D = \frac{1}{2}$. In the following we restrict our attention to the interesting situation $D < 1$; in this case the tunneling amplitude γ (one sets $\gamma_1 + i\gamma_2 = \gamma e^{i\theta/2}$) flows to SC under the renormalization group and reaches the value 1 at an energy scale $T_{\rm B} \sim \gamma^{1/(1-D)}$: below this scale, the system enters the SC regime (Fig.1).

The physics in the SC regime is governed by the SC fixed point, that is reached at vanishing energy $\frac{E}{T_{\rm B}} \to 0$. Physically, this fixed point corresponds to a perfect hybridization of the wires: the IRLM at the SC fixed point can be described in terms of *new* fermions $\Psi_a(x,t)$, the SC fields, that do not feel at all the impurity: the system is homogeneous, with Hamiltonian $H_0^{\rm SC} = H_0[\Psi_1] +$

$H_0[\Psi_2]$. There is a simple relationship between the SC fields Ψ_a and the original physical fields ψ_a: for incoming fields (i.e. in the region $x<0$) their densities are the same, $:\psi_a^\dagger\psi_b:\,(x<0,t)=:\Psi_a^\dagger\Psi_b:\,(x<0,t)$. For outgoing fields (region $x>0$), the relationship is still linear, $:\psi_a^\dagger\psi_b:\,(x>0,t)=\mathrm{B}_{ab}^{cd}\;:\Psi_c^\dagger\Psi_d:\,(x>0,t)$ with $\mathrm{B}(D,\theta)$ a 4×4 matrix that can be worked out [35].

(i) Out-of-equilibrium SC fixed point: linear regime. We first study the SC fixed point (i.e. we set $T_\mathrm{B}=\infty$), and we force the system out-of-equilibrium by considering a bath with thermodynamical variables (μ_a,T_a) that in turn specifies the density matrix for incoming states in wire a. Since densities of the original and SC incoming fields coincide, and since SC fields Ψ are right-moving fields of an homogeneous system, we can prove that the density matrix at the SC fixed point *out of equilibrium* is $\rho^{\mathrm{SC}}_{\mathrm{N.Eq}}=\rho^{\mathrm{SC}}_{\mathrm{N.Eq},1}\otimes\rho^{\mathrm{SC}}_{\mathrm{N.Eq},2}$, with :

$$\rho^{\mathrm{SC}}_{\mathrm{N.Eq},a}=e^{-\frac{1}{T_a}[H_0^{\mathrm{SC}}[\Psi_a]-\int_{-\infty}^{\infty}dx\,\mu_a:\Psi_a^\dagger\Psi_a:(x)]}. \qquad (3)$$

In other words, the Hershfield's operator [28] at the SC fixed point is $\int dx\,\mu_a:\Psi_a^\dagger\Psi_a:(x)$. This seemingly simple operator becomes a complicated, non-local object that mixes the wires once reexpressed in terms of the physical fields ψ_a. Our Eq. (3) generalizes an expression obtained earlier for $\theta=\pi/2$, $\mu_a=0$ and $T_1\neq T_2$ [36] [43]. Besides, we generalize this approach for time-dependent forcing $\mu_a(t)$, where $\mu_a(t)$ is an effective chemical potential close to the impurity [44]. Notice that as far as properties close to the impurity are concerned, one can equivalently couple system to a space-varying chemical potential and replace $\mu_a(t)\to\mu_a(t-x)$ [45]. Those "right-moving"potentials preserve the holomorphic dependence of the fields, and can be absorbed exactly by a gauge transformation $\Psi_a(z)\to\mathscr{U}^{-1}_{\mathrm{N.Eq}}(z)\cdot\Psi_a(z)=e^{\Xi_a(z)}\Psi_a(z)$ where $\Xi_a(x,t)=i\int_0^{t-x}dt'\,\mu_a(t')$ is continuated to the complex plane $z=i(t-x)$. The gauge transformation maps the out-of-equilibrium theory onto an equilibrium one : for a generic operator built with fermions fields, one has $\big\langle\mathcal{A}[\Psi]\big\rangle_{\mathrm{N.Eq}}=\big\langle\mathcal{A}[\Psi^{\mathrm{eff}}]\big\rangle_{\mathrm{Eq}}$, with $\Psi^{\mathrm{eff}}=\mathscr{U}_{\mathrm{N.Eq}}\cdot\Psi$ and $\big\langle\,\cdot\,\big\rangle_{\mathrm{Eq}}$ is the equilibrium average with density matrix $e^{-\frac{H_0(\Psi_a)}{T_a}}$. The explicit form of this transformation reads:

$$\mathscr{U}_{\mathrm{N.Eq}}(z)=\mathcal{R}e^{-i\oint_z dw\,\Xi_a(w):\Psi_a^\dagger\Psi_a:(w)} \qquad (4)$$

where $\mathcal{R}e^{\oint_z d\omega\hat{X}(\omega)}=1+\sum_{k>0}\frac{1}{k!}\oint_{\mathcal{C}_1}dz_1...\oint_{\mathcal{C}_k}dz_k\,X(z_1)...X(z_k)$ is an ordered exponential with the contours $\mathcal{C}_k$ radially ordered around z, $|z_1-z|>...>|z_k-z|>0$. Our Eq. (4) proves that forcing the system out-of-equilibrium at the SC fixed point thus amounts deformation of the underlying conformal field theory describing the SC fixed point. This results holds irrespective of the nature of SC fixed fixed point – including non Fermi liquid fixed points.

Using $\mathscr{U}_{\mathrm{N.Eq}}$, let us now evaluate the electrical current operator on wire a close to the impurity, $I_a=:\psi_a^\dagger\psi_a:\,(0^-)\,-\,:\psi_a^\dagger\psi_a:\,(0^+)$. $\mathscr{U}_{\mathrm{N.Eq}}$ mixes descendent fields amongst each other, and we find that the current acquires a part proportional to the identity: $\hat{I}_a^{\mathrm{eff}}(t)=\sum_{\mu=0}^{3}\Lambda_a^{bc}(t)\,:\Psi_a^\dagger\Psi_b:\,(0,t)+\frac{\sin^2\theta}{2\pi}V(t)$ with $V=\mu_1-\mu_2$ the voltage across the impurity. One thus recovers the linear regime $I_a^{\mathrm{SC}}=\langle\hat{I}_a\rangle_{\mathrm{N.Eq}}=G_0\,V(t)$ with $G_0=\sin^2\theta\,\frac{e^2}{h}$.

(ii) Around the SC fixed point. We now consider finite values of $\frac{\mu_a}{T_\mathrm{B}}$, $\frac{T_a}{T_\mathrm{B}}$, that drive the system away from the SC fixed point. The inhomogeneity of the system reappears: the SC fields Ψ_a now undergo scattering processes at $x=0$, and we should consider all irrelevant processes allowed by symmetries. The IRLM is a Fermi liquid, so the lowest order process is an energy-momentum tensor, call it $\mathcal{O}_2$, with coupling $\propto\frac{1}{T_\mathrm{B}}$. It determines the back-scattered current $I_{\mathrm{BS}}=I_a^{\mathrm{SC}}-\langle\hat{I}_a\rangle\propto V^3/T_\mathrm{B}^2$ at lowest order: this is Fermi Liquid Theory. At higher orders T_B^{-n} (e.g. to compute the first finite temperature correction $\propto T^2V^3/T_\mathrm{B}^4$ to I_{BS}), $\mathcal{O}_2$ is not sufficient and independent, higher order processes must be considered. In general one faces here several difficulties, that prevent any practical calculations: the number of independent processes (and corresponding couplings) grows very rapidly with n ; moreover there is *no* way to fix the couplings [46]. Therefore, physical quantities depend on a whole series of energy scales (the couplings are dimension-full) that are generated by the finite distance to the SC fixed point. On the contrary, as we shall see now, in an integrable system, out-of-equilibrium physical quantities depend on a *single* energy scale T_B, and are therefore given by universal scaling functions.

The IRLM is integrable, and its infinity of conserved quantities $\mathcal{O}_{2n}$ highly constraints the allowed processes. It has been shown [37] that a dual description of the Hamiltonian can be derived: $H=H_0^{\mathrm{SC}}+H_\mathrm{B}^{\mathrm{SC}}$ with $H_\mathrm{B}^{\mathrm{SC}}=\sum_{n=1}^{\infty}\frac{g_{2n}}{T_\mathrm{B}^{2n-1}}\mathcal{O}_{2n}(x=0)$ where the numbers g_{2n} are known exactly. This dual Hamiltonian has been used to derive equilibrium properties [35, 38] in the SC regime. The operators $\mathcal{O}_{2n}$ are the only independent allowed processes, and transfer *integer* charges across the impurity: we call such a system a "super Fermi liquid".

We now evaluate a general out-of-equilibrium $N-$point correlators of the currents. We perform a Keldysh expansion in powers of T_B^{-1}, starting at time $t=-\infty$ at the *out-of-equilibrium* SC fixed point ($\lambda_\mathrm{B}=0$) and adiabatically turning on $\lambda_\mathrm{B}(t)$ to a finite value $\lambda_\mathrm{B}(0)=\frac{1}{T_\mathrm{B}}>0$. Using a super-operator formulation [39], and the fact that in a super Fermi liquid, nested commutators of $H_\mathrm{B}^{\mathrm{SC}}$ with the currents can be represented as contour integrals [47], we arrive at the following form for the super-operator that implements the Keldysh expansion:

$$\mathscr{U}_\mathrm{B}=\mathcal{R}e^{-\oint_z dw\,H_\mathrm{B}^{\mathrm{SC}}(w)}. \qquad (5)$$

Importantly, short-distance divergences in the expansion are exactly canceled by the regularization (point splitting) inherited from integrability, even out-of-

equilibrium. This proves formula (1), an expression that is finite order by order. For incoming fields, we find that $\mathscr{U}_{\rm B}\cdot\Psi_a(x<0)=\Psi_a(x<0)$ as expected from causality.

(iii) Predictions. Calculating the effective current operator $\hat{I}^{\rm eff}=\sum_{n=0}^{\infty}\frac{\hat{I}^{(n)}}{T_{\rm B}^n}$ by exploiting (1) in a perturbative way requires the computation of expressions $\mathscr{U}_{\rm N.Eq}\cdot\mathcal{O}_{2n_1}\star\ldots\star\mathcal{O}_{2n_k}\star:\Psi_a^\dagger\Psi_b:$ where $\mathcal{O}\star\Phi(z)\equiv\oint_z dw\mathcal{O}(w)\Phi(z)$. It becomes a purely algebraic problem: at order $T_{\rm B}^{-n}$, $\hat{I}^{(n)}$ lives in the space $\mathcal{E}_{n+1}$ of Kac Moody descendent fields [40] with conformal weights $\leq n+1$ [48]. At large order, we resort to a Mathematica code (a calculator in $\mathcal{E}_{n+1}$, a space with intricate structure, in which basic operations are neither commutative nor associative). We performed

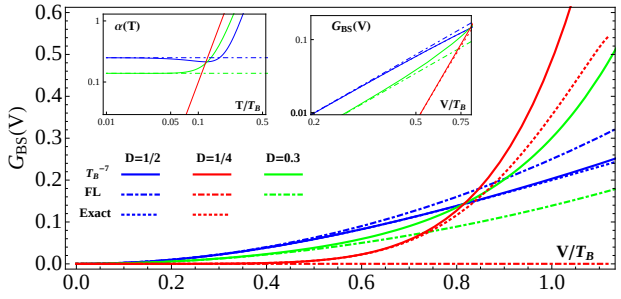

FIG. 2: [Color online] Differential conductance $G_{\rm BS}(V)$ at $T=0,\theta=\frac{\pi}{2}$ in $\frac{e^2}{h}$ units. We also plot the exact results known at $D=\frac{1}{2}$ (free fermions) and $D=\frac{1}{4}$ (self-dual point). Dotted lines are the FL results (up to $T_{\rm B}^{-2}$). The inset shows the temperature dependence of α, the first non-linearity in $\frac{G_{\rm BS}}{G_0}=\alpha(T)\frac{V^2}{T_{\rm B}^2}+\mathcal{O}(V^4)$.

calculations up to order $T_{\rm B}^{-7}$, and we give here the expression of the first terms of the backscattered current $I_{\rm BS}=I_0^{\rm SC}-\langle\hat{I}\rangle$ for a static applied bias V, at $T\neq0$ and in the resonant case $\epsilon_d=0$ (from now on one considers the symmetrized current $I=\frac{I_1-I_2}{2}$): $\frac{I_{\rm BS}}{I_0^{\rm SC}}=\frac{X(V^2+(2\pi T)^2)^2}{48D^2T_{\rm B}^2}+\frac{X(V^2+(2\pi T)^2)}{(12T_{\rm B})^2}\left(\frac{3g_4(2(2\pi T)^2X(X-15)+3V^2(X^2-10X+5))}{160D^3T_{\rm B}^2}-\frac{2(2\pi T)^2(1+5X+X^2)+3V^2(X^2+1)}{160D^4T_{\rm B}^2}\right)+\mathcal{O}(\frac{1}{T_{\rm B}^6})$ with $X=4D-1$ and $g_4=\frac{D}{6\pi^2}\frac{\Gamma(\frac{6}{3-X})\Gamma(\frac{X+1}{6-2X})^3}{\Gamma(\frac{3(X+3)}{6-2X})\Gamma(\frac{2}{3-X})^3}$. The resulting backscattered non-linear conductance $G_{\rm BS}(V)=\frac{\partial I_{\rm BS}}{\partial V}$ is shown in Fig.2. We see the excellent agreement (up to $0.7\,T_{\rm B}$) with the exact expressions available in the free and self-dual cases: it improves on the FL results (drastically close to the self-dual point $D=\frac{1}{4}$).

We then consider an AC bias $\mu_{1/2}(t)=\pm V/2\cos\omega t$. In the small voltage regime, the current can be written in terms of the admittance $\underline{Y}^{(1)}(\omega)=(G+i\omega C)(\omega)$ (C is the capacitance), such that $I(t)=V\,\Re(\underline{Y}^{(1)}e^{i\omega t})$, and non-linear effects now lie in the ω-dependence. We

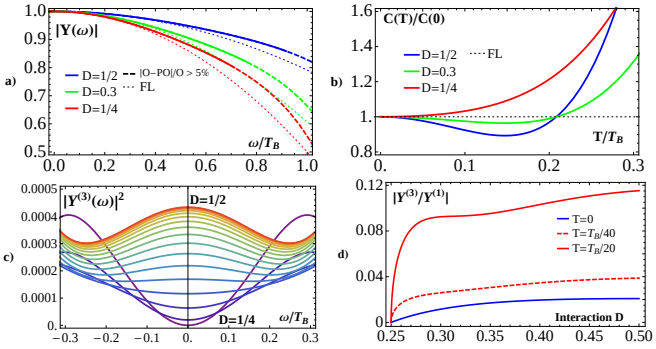

FIG. 3: [Color online] ($\theta=\frac{\pi}{2}$) (a) Modulus of the admittance $|\underline{Y}(\omega)|$ for an AC bias (in e^2/h units). The curves are dashed when the last two orders of our expansion differ by more than 5%. (b) Capacitance $C(\omega=0,T)$, normalized by the FL value $C(0,0)$. (c) Amplitude of the 3ω harmonics $\underline{Y}^{(3)}$ as a function of frequency ($T=0$) (d) Influence of temperature on the efficiency of harmonics production.

show in Fig.3 the amplitude $|\underline{Y}^{(1)}|$: it is maximal in the non-interacting case. On top of the $T,\omega\to0$ limit $C(0,0)=\frac{e^2\cos2\theta}{4DT_{\rm B}h}$ predicted by FL theory, we find a rich dependence on ω and T, in particular $C(\omega=0)$ is significantly affected by temperature in a way that depends on the interactions.

We are also able to compute explicitely higher harmonics of the current. We define higher admittances as Fourier coefficients $\underline{Y}^{(n)}(\omega)=\lim_{V\to0}\frac{\omega T_{\rm B}^{n-1}}{\pi V^n}\int_{-\pi/\omega}^{\pi/\omega}I(t)e^{-in\omega t}dt$. In the particle-hole symmetric case $\epsilon_d=0$, the current is odd in V and the first non linear contribution is an odd harmonic at 3ω. In Fig.3 we show how frequency affects the amplitude $\underline{Y}^{(3)}$. The FL result $|\underline{Y}^{(3)}_{FL}|=\frac{(4D-1)\sin^2\theta}{192D^2}$ cannot capture the sign change of the ω dependence when interactions are switched on. If we break particle-hole symmetry with a grid potential $\epsilon_d d^\dagger d$, $2\omega-$harmonics are generated, and the lowest contribution to the admittance reads $\underline{Y}^{(2)}=-ig_4\frac{3\cos\theta}{64D^2}\frac{\epsilon_d\omega}{T_{\rm B}^2}$: it does *not* depend on the first process $\mathcal{O}_2$, and is thus missed by the FL approach.

Conclusions. We have shown that out-of-equilibrium QIS at their SC fixed point can be described by a simple deformation of the underlying conformal field theory describing the equilibrium SC fixed point, in the same spirit as finite temperature can be obtained as a mapping from the plane to the cylinder geometry. Then, in integrable QIS that are super Fermi liquids, we have shown that the integrable structure can be used to build a super-operator mapping the out-of-equilibrium interacting theory onto a free one at equilibrium, and allowing us to show that physical properties are universal. This also results in the possibility to obtain controlled, exact analytical results in the whole SC regime for the expansion of the universal scaling function of the currents. We have applied this method to the IRLM, revealing some non-linear effects

that are not described by the usual Fermi liquid theory. Our approach furnishes a mean to investigate in a controlled way many other out-of-equilibrium properties of super Fermi liquids, like (finite frequency) noise, full counting statistics, or non-linear thermal transport. It also raises the question whether it can be generalized to non (super) Fermi liquid integrable QIS.

Acknowledgements: E.B. would like to thank S. Florens, O. Parcollet, I. Paul, T. Popov and H. Saleur for stimulating discussions and encouragements. L.F. acknowledges support from the French DGA.

[1] D. Goldhaber-Gordon *et al.*, Nature **391** (1998) 156; D. Goldhaber-Gordon *et al.*, Phys. Rev. Lett. **81** (1998) 5225.

[2] R. Hanson, L. Kouwenhoven, J. Petta, S. Tarucha, and L. Vandersypen, Rev. Mod. Phys. **79** (2007)1217.

[3] A. Nitzan, and M. A. Ratner, Science **300** (2003) 1384.

[4] M. A. Reed *et al.*, Science **278** (1997) 252 ; W. Liang, M. Shores, M. Bockrath, J. Long, and H. Park Nature **417** (2002) 725.

[5] E. Scheer, P. Joyez, D. Esteve, C. Urbina, M. Devoret, Phys. Rev. Lett. **78** (1997) 3535.

[6] J. Cardy, and D.C. Lewellen, Phys. Lett. B **259** (1991) 274.

[7] I. Affleck, Nucl. Phys. B **336** (1990) 517.

[8] I. Affleck, and A.W.W. Ludwig, Nucl. Phys. B **352** (1991) 849.

[9] P. Nozières, J. Low Temp. Phys. **17** (1974) 31.

[10] H. R. Krishna-murthy, J. W. Wilkins, and K. G. Wilson, Phys. Rev. B **21** (1980) 1003 and Phys. Rev. B **21** (1980) 1044.

[11] A. Branschädel, E. Boulat, H. Saleur, and P. Schmitteckert, Phys. Rev. Lett. **105** (2010) 146805.

[12] L. Mühlbacher, and E. Rabani, Phys. Rev. Lett. **100** (2008) 176403 ; M. Schiró, and M. Fabrizio, Phys. Rev. B **79** (2009) 153302 ; P. Werner, T. Oka, and A.J. Millis, Phys. Rev. B **79** (2009) 035320 and Phys. Rev. B **81** (2010) 035108.

[13] F. Anders, and A. Schiller, Phys. Rev. Lett. **95** (2005) 196801.

[14] S. Andergassen, M. Pletyukhov, D. Schuricht, H. Schoeller, and L. Borda, Phys. Rev. B **83** (2011) 205103.

[15] R. Landauer, IBM J. Res. Dev. **1** (1957) 223.

[16] M. Büttiker, Phys. Rev. Lett. **57** (1986) 1761.

[17] Y. Imry, and R. Landauer, Rev. Mod. Phys. **71** (1999) S306.

[18] A. Schiller, and S. Hershfield, Phys. Rev. B **51** (1995) 12896.

[19] A. Schiller, and S. Hershfield, Phys. Rev. B **58** (1998) 14978.

[20] E. Sela, and I. Affleck, Phys. Rev. Lett. **102** (2009) 047201.

[21] E. Sela, and I. Affleck, Phys. Rev. Lett. **103** (2009) 087204.

[22] V. Bazhanov, S. Lukyanov, A. Zamolodchikov Nucl. Phys. B **549** (1999) 529.

[23] P. Fendley, A.W.W. Ludwig, and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005.

[24] P. Fendley, A.W.W. Ludwig, and H. Saleur, Phys. Rev. B **52** (1995) 8934.

[25] E. Boulat, H. Saleur, and P. Schmitteckert, Phys. Rev. Lett. 101, 140601 (2008).

[26] A. Chubukov, and D. Maslov, Phys. Rev. B **69** (2004) 121102.

[27] A. Chubukov, D. Maslov, and A. Millis, Phys. Rev. B **73** (2006) 045128.

[28] S. Hershfield , Phys. Rev. Lett. **70** (1993) 2134.

[29] M. Büttiker, A. Prêtre, and H. Thomas, Phys. Rev. Lett. **70** (1993) 4114.

[30] A. Jauho, N. Wingreen, and Y. Meir, Phys. Rev B **50** (1994) 5528.

[31] P. Wiegmann, and A. M. Finkelstein, Sov. Phys. JETP **48** (1978) 102.

[32] B. Doyon, Phys. Rev. Lett. **99** (2007) 076806.

[33] See e.g. S. Carr, D. Bagrets, and P. Schmitteckert, Phys. Rev. Lett. **107** (2011) 206801 ; M. Goldstein, Y. Weiss, and R. Berkovits, Physica E **42** (2010) 610 ; D. Kennes, and V. Meden, Phys. Rev. B**85** (2012) 245101 ; D. Kennes, and V. Meden, Phys. Rev. B **87** (2013) 075130 ; F. Guettge, F. Anders, U. Schollwoeck, E. Eidelstein, and A. Schiller, Phys. Rev. B **87** (2013) 115115.

[34] I. Affleck, Acta Phys. Polon. B**26** (1995),1869.

[35] E. Boulat, and H. Saleur, Phys. Rev. B **77** (2008) 033409.

[36] D. Bernard, and B. Doyon, J. Phys. A: Math. Theor. **45** (2012) 362001.

[37] F. Lesage, and H. Saleur, Nucl. Phys. B **546** (1999) 585.

[38] L. Freton, E. Boulat, and H. Saleur, preprint `arXiv:1301.6535` (2013).

[39] U. Harbola, and S. Mukamel, Phys. Rep. **465** (2008) 191, and references therein.

[40] P. Di Francesco, P. Mathieu, and D. Sénéchal, "Conformal field theory", New York, Springer (1997).

[41] D. Bernard, and B. Doyon, preprint `arXiv:1302.3125` (2013).

[42] This is in agreement with the common wisdom that the "largest energy scale E cuts the RG flow".

[43] During the redaction of this manuscript, we became aware of an independent construction of the non-equilibrium density matrix *at* the SC fixed point [41] for time-independent forcing.

[44] In general, this chemical potential differs from the one imposed far away on the electrodes, see [29].

[45] The spatial dependence we introduce by this trick is generically extremely weak in the SC regime $\omega \lesssim T_{\rm B}$, with a wavelength $\gtrsim 1$ m for $T_{\rm B} \lesssim 100$K.

[46] Only the coupling of the first operator, $\mathcal{O}_2$, can be fixed, since it effectively amounts to a definition of $T_{\rm B}$.

[47] We use the crucial property that the operator product expansion $H_{\rm B}^{\rm SC}(z)A(w) = \sum_k \frac{\{H_{\rm B}^{\rm SC}A\}_k(w)}{(z-w)^k}$ only has integer exponents k (i.e. $H_{\rm B}^{\rm SC}$ and A are mutually local) if A is a current operator, and this property holds recursively, i.e. replacing A by $\{H_{\rm B}^{\rm SC}A\}_k$ etc.

[48] The dimension of $\mathcal{E}_n$ grows as $\sim n^{-\frac{1}{2}} e^{2\pi\sqrt{\frac{n}{3}}}$.